Self Assessment Test
for the International Masters Programme
"Physics of Life, Light & Matter"

Institute of Physics
University of Rostock
Germany

At the Institute of Physics we like happy students. Of course our high quality education in physics requires commitment and also a lot of hard work. The happiness comes from the great reward of a deeper understanding of how nature works, i.e being a successful student. Think of of a bicycle downhill race, which is great fun unless you crash. A crash seems to be more likely for somebody who cannot even ride in a straight line on a flat road. This test is designed for you to assess for yourself whether you can drive basic physics in a straight line. Those who have no troubles whatsoever solving this test are well equipped for the fun race of our Master's programme. Those who are unable to grasp the better part of the test are likely to crash.

This test takes time. A few hours are needed to complete it thoroughly. Some tasks take more time than others. Some tasks are easier than others. Feel free to use a textbook, when you need to brush up on some of the details. We picked problems from the courses of our Bachelor's programme in theoretical physics (T1-T6) and experimental physics (E1-E6)

If you are not sure based on your own judgement about the your performance, you are invited to send your solution the the email address you find on the website where you found this test. We will have a look at it and provide you with the solutions. This will have no consequences at all towards your eligibility for the programme.

Have fun!

Calculus (T1)

A1 Divergence theorem

Let the vector field $\mathbf{F} = (x^2, y^2, z^2)^{\mathrm{T}}$ be given.

a) Calculate the integral $\oiint \mathbf{F} \cdot \mathbf{n} dS = \oiint \mathbf{F} \cdot \mathbf{dS}$ over the surface area of a cube of unit length located in the first octant of the cartesian coordinate system.

b) Calculate the same integral by using the divergence theorem.

A2 Harmonic Oscillator

The equation of motion of a one-dimensional driven damped harmonic oscillator can be written as

$$\ddot{x}(t) + \gamma \dot{x}(t) + \omega_0^2 x(t) = f(t)\,.$$

The driven force acts as an impulse $f(t) = f\delta(t)$.

a) Show that the Fourier transforms of the first and second derivatives $g(t) = \dot{x}(t)$ and $h(t) = \ddot{x}(t)$ are $G(\omega) = i\omega X(\omega)$ and $H(\omega) = -\omega^2 X(\omega)$, respectively, and show the validity of the relation $\mathcal{F}[\delta(t)] = 1$.

b) Transform the equation of motion into Fourier space and obtain the spectrum $X_1(\omega) = \mathcal{F}[x(t)]$.

c) Calculate the Fourier transform $X_2(\omega)$ of the function

$$x(t) = f\Theta(t)\frac{\sin(\bar{\omega}t)}{\bar{\omega}}\mathrm{e}^{-\frac{\gamma}{2}t}$$

with $\bar{\omega}^2 = \omega_0^2 - (\gamma/2)^2$ and show that both spectra are identical, $X_2(\omega) = X_1(\omega)$.
Hint: Use the decomposition $\sin x = \frac{1}{2i}(\mathrm{e}^{ix} - \mathrm{e}^{-ix})$.

d) Plot the solution $x(t)$.

Theoretical Mechanics (T2)

A3 Inclined plane

A point mass m slides frictionless along an inclined plane (inclination angle α). At time $t = 0$, the point mass is at rest at some position $\vec{r}_0$ on the plane.

a) Derive the equation of motion of the point mass in Cartesian coordinates using d'Alembert's principle.

b) Derive the equation of motion in suitable coordinates using the Lagrange equations of the second kind.

c) Solve the obtained differential equations from parts a) and b) with the given initial conditions.

A4 Satellite orbit

A point mass m moves in the gravitational potential of the Earth with constant angular velocity $\vec{\omega} = \omega\vec{e}_z$ on a circular orbit of radius R.

a) Formulate und classify the constraints on the motion of the point mass.

b) Write down the Lagrange function (i.e. the Lagrangian) in suitable coordinates. Determine the generalised momentum. Derive the Hamilton function using a Legendre transformation.

c) Formulate the canonical equations (i.e. Hamilton's equations). Derive from them the equation of motion for the point mass and solve it. What is the orbital period of the point mass?

Electrodynamics and Optics (T3)

A5 Electromagnetic waves

The following electric field (E_x and E_y constant) is given in carthesian coordinates:

$$\vec{E}(z,t) = E_x \cos(kz - \omega t)\vec{e}_x + E_y \sin(kz - \omega t)\vec{e}_y, \quad \omega = c_0 k,$$

with c_0 the vacuum speed of light.

a) Write down the general Maxwell equations with charges and currents. How do they simplify in vacuum? Compute the magnetic field $\vec{B}(z,t)$ corresponding to $\vec{E}(z,t)$, and show that the fields fulfill Maxwells equations in vacuum. *Hint: All integration constants are set to zero!*

b) Calculate the energy density $u(z,t)$ and the Poynting vector $\vec{S}(z,t)$ of the fields. Check the energy conservation for this field.

c) What is the polarization of this field?

A6 Electrostatics

A static, continuous, spherical-symmetrical charge distribution is given as

$$\varrho(r) = \varrho_0 \frac{\exp(-\lambda r)}{r}, \quad \lambda > 0.$$

a) Compute the full charge Q for this distribution and write down ϱ_0 in terms of Q and λ.

b) Compute the electric field $\vec{E}(r) = E(r)\vec{e}_r$, by directly integrating the corresponding Maxwell equation. *Hint: Use* $\nabla \cdot \vec{E}(r) = \frac{1}{r^2}\frac{d}{dr}\left[r^2 E(r)\right]$ *and partial integration.*

c) Derive the approximate solution for $\lambda r \gg 1$ and compare to the field of a point charge.

Quantum Physics (T4)

A7 Wave mechanics

At a given time a particle moving along the x-axis is described by the wave function $\psi(x)$, which reads as

$$\psi(x) = \mathcal{N} e^{-\frac{x^2}{2\sigma^2}}, \quad \sigma > 0.$$

a) Determine the normalization constant $\mathcal{N}$.

b) Calculate the expectation value and the standard deviation of position operator $\hat{x}$ and its momentum $\hat{p}_x$. *Hint: Use the following relations*
$\int_{-\infty}^{\infty} dx\, e^{-ax^2} = \sqrt{\frac{\pi}{a}}, \quad \int_{-\infty}^{\infty} dx\, x^2 e^{-ax^2} = \frac{1}{2a}\sqrt{\frac{\pi}{a}}$, *with* $Re(a) > 0$.

c) Name the commutator of position and momentum. Write down the formula for the Heisenberg uncertainty principle for two general operators, and specifically for position and momentum. Does $\psi(x)$ describe a state of minimal position-momentum uncertainty?

A8 Dirac-notation and Measurement

A quantum-mechanical system at time $t = 0$ is given by the state

$$|\psi(0)\rangle = \frac{1}{\sqrt{5}}(0|\phi_{-1}\rangle + 1|\phi_0\rangle - 2i|\phi_{+1}\rangle),$$

where $|\phi_{-1,0,1}\rangle$ are orthonormalized eigenstates of a Hamilton-Operator $\hat{H} = \sum_{j=-1}^{1} E_j |\phi_j\rangle\langle\phi_j|$ with pairwise distinct eigenvalues.

a) Give the explicite form for $|\psi(t)\rangle$. What is the probability at $t \geq 0$ to measure the energy value E_i ($i = -1, 0, +1$)?

b) An operator $\hat{S} = \hbar\Omega(|\phi_{-1}\rangle\langle\phi_{+1}| + |\phi_{+1}\rangle\langle\phi_{-1}|)$ is given. Its eigenstates are $|\psi_\pm\rangle = \frac{1}{\sqrt{2}}(|\phi_{-1}\rangle \pm |\phi_{+1}\rangle)$ and $|\phi_0\rangle$. Compute the corresponding eigenvalues $S_\pm$ and S_0.

c) At a time $t_1 > 0$ the measurement of $\hat{S}$ yields S_-. Give the quantum state immediatly after the measurement. What are the probabilities for the energy values E_i in a subsequent measurement?

d) What is the density operator $\hat{\varrho}(t)$ for $|\psi(t)\rangle$, $0 < t < t_1$? How do you determine an expectation value of an operator $\hat{A}$ with the density operator?

Thermodynamics (T5)

A9 Carnot Cycle

Calculate the efficieny of a Carnot machine operated with an ideal gas. Use the following thermodynamic process in the $p - V$ diagram (the closed cycle is composed of two isotherms and two adiabats):

a) adiabatic compression $p(T_1, V_1) \to p(T_2, V_2)$,

b) isothermal expansion $p(T_2, V_2) \to p(T_2, V_3)$,

c) adiabatic expansion $p(T_2, V_3) \to p(T_1, V_4)$,

d) isothermal compression $p(T_1, V_4) \to V(T_1, V_1)$.

Hint: The result is $\eta = 1 - T_1/T_2$ with $T_1 < T_2$.

Statistical Physics (T6)

A10 Ideal Gas

a) Show for a classical ideal gas with the Hamilton function

$$H_N(p, q) = \sum_{i=1}^{N} \frac{\vec{p_i}^2}{2m} \quad ,$$

that the canonical partion function is given by

$$Z(T, V, N) = \frac{1}{N!h^{3N}} \int \mathrm{d}^{3N}\vec{q}\,\mathrm{d}^{3N}\vec{p}\, e^{-\beta H_N(p,q)} = \frac{V^N}{N!\lambda^{3N}} \quad ,$$

where $\beta = 1/k_BT$ and $\lambda = \sqrt{h^2/2\pi m k_B T}$ is the thermal wavelength. Use the transition to the continuum:

$$\sum_i w_i \to \frac{1}{h^{3N}N!} \int d^{3N}r\; d^{3N}p\,.$$

b) Calculate the free energy $F = -k_BT \ln Z(T, V, N)$. Use Stirling's formula

$$\ln N! \approx N(\ln N - 1)$$

which is valid for large values of N.

c) Using the expression for the free energy, calculate the pressure p, the chemical potential μ, the entropy S, the internal energy U, and the heat capacity C_V in dependence of T, V und N. Notice: The results are the ideal gas expressions known from Thermodynamics.

Mechanics and Heat (E1)

A11 Tree swing

Far off you see a tree with a children's swing fixed at half height and reaches the ground. The swing oscillates because of the wind. You notice, the swing turns at the outer right position every 5 seconds. What ist the height of the tree?

A12 Cube on the water

A cube with the edge length of 10 cm is floating on water ($\rho = 1$ g cm^{-3}) and half of its height looks still out of the water.

a) Calculate the average density of the cube.

b) Calculate the work which is necessary to drop the cube completely under water.

c) Calculate the frequency of the undamped oscillation which is observable if the cube is released.

A13 Mercury Column

A mercury column (length: $l = 40\,\mathrm{cm}$, viscosity: $\eta = 15.7\,\mathrm{Pa\cdot s}$, density: $\rho = 13.6\,\mathrm{g\cdot cm^{-3}}$) oscillates in a U-shaped glass pipe (inner diameter $d = 5.0\,\mathrm{mm}$).

a) Find the equation of motion!

b) Determine the decay time τ , the angular frequency ω and the logarithmic decrement δ of the oscillation.

c) How long does it take for the amplitude to reach exactly half of its initial value? How many oscillation periods occur during this time?

Electricity, Magnetism and Optics (E2)

A14 Plate Capacitor

A plate capacitor has a plate area of $A = 1500\,\text{cm}^2$ and a gap of $d = 2.5\,\text{mm}$ between the plates. It is charged until the voltage across the plates reaches $U = 800\,\text{V}$.

a) Determine the capacitance and the electric energy stored in the capacitor.

b) Determine the attractive force between the plates.

A15 Induction Cooking

Induction cooking heats the cooking vessel by magnetic induction. The magetic field is created by a work coil beneath the surface of the cooker. The work coil together with a fixed capacitor usually forms a resonant tank circuit with a resonance frequency of roughly 100 kHz.

a) Name the fundamental heating processes.

b) Why does induction cooking work better with iron/steel cookware compared to copper/aluminium?

c) Why does the resonance frequency of the tank circuit increase with a copper pot on top?. Why does the resonance frequency of the tank circuit decrease with an iron pot on top?

A16 Photoelectron

Vacuum ultraviolet photons with a wavelength of 50 nm hit a metal surface with a work function of 4 eV.

a) How large is the energy of the photons?

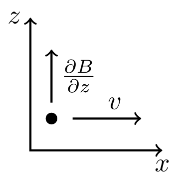
b) What is the maximal speed of the electrons and how large is the de Broglie wavelength of these electrons?

c) Assume the electrons hit a slit with a width of 100 nm. At which deflection angle do you expect the first diffraction minimum?

Atomic and Molecular Physics (E4)

A17 Stern-Gerlach Experiment

At the Stern-Gerlach experiment, a beam of silver atoms in the ground state ($5^2S_{1/2}$, atomic mass $m = 1.8 \cdot 10^{-25}$ kg) with a velocity in x-direction of $v = 750$ m/s is deflected by a strong inhomogeneous magnetic field, oriented perpendicular to the direction of the moving atoms. The gradient of the magnetic field is given by $\frac{\delta B}{\delta z} = 1.4$ T/mm. The magnetic field covers a distance of $l = 3.5$ cm in x-direction, with a detection screen being located straight behind the magnetic field.



a) Please, explain the separation of the atomic beam into two beams inside the magnetic field

b) Please, calculate the distance d between both beams at the detection screen, if the magnetic moment of the silver atoms is given by $\mu = 1 \cdot 10^{-23}$ J/T.

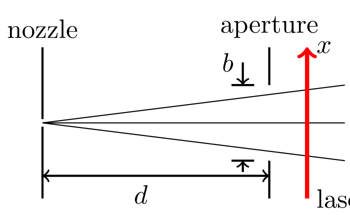
A18 Doppler broadening

Consider the absorption and emission of light at the Lyman-α line of Hydrogen atoms at room temperature (300 K).

a) What are the wavelength and frequency of the Lyman-α line?

b) Determine the line width caused by Doppler broadening.

c) Consider a beam of H-atoms emerging a 50 μm nozzle pointing at an aperture with a width of $b = 1\,\text{mm}$ at a distance of $d = 10\,\text{cm}$. Behind the aperture a tunable monochromatic laser is sent through the atoms perpendicularly in x-direction. Determine the residual Doppler width of the absorption line.



Hint: Recall the equilibrium velocity distribution (Maxwell-Boltzmann) in one direction (x) for particles in a gas at Temperature T : $n(v_x)\mathrm{d}v_x = \frac{N}{v_0\sqrt{\pi}} e^{-(v_x/v_0)^2}\mathrm{d}v_x$, with the number of particles per volume $n(v_x)\mathrm{d}v_x$ in the velocity interval $[v_x, v_x + \mathrm{d}v_x]$, the most probable velocity $v_0 = \sqrt{2k_\mathrm{B}T/m}$, the total number of particles per volume N, the particle mass m.

Solid State Physics (E5)

A19 Elastic scattering on a crystal lattice (Ewald construction)

Explain the principle of the Ewald construction in two dimensions (use a drawing). Let's say we have a neutron scattering experiment on a big single crystal. Use your sketch for the Ewald construction to clarify, under which circumstances the scattered neutron is scattered a second time before leaving the crystal.

A20 Bloch's theorem

What is the cause of Bloch's theorem for crystal electrons and what is one formulation of this theorem?

Nuclear and Particle Physics (E6)

A21 Nuclear Fission

Estimate by means of the semi-empirical Bethe-Weizsäcker mass formula the energy release upon fission of ^{235}U.

a) Assume (nearly) symmetric fragments.

b) How much energy is released upon induced fission by a thermal neutron: $^{235}\mathrm{U} + \mathrm{n} \rightarrow\ ^{98}\mathrm{Sr} + ^{136}\mathrm{Xe} + 2\mathrm{n}$?